JOURNAL OF APPROXIMATION THEORY 66, 109–113 (1991)

Note

A Remark on Hermite–Lagrange Interpolation

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Communicated by T. J. Rivlin

Received April 29, 1989

A simple proof of a recent result of G. Berger and M. Tasche concerning the coefficients of the Hermite–Lagrange 2-point interpolation polynomial and uniform approximation by such polynomials is given. This provides an easy access to a number of consequences of their result which have attracted considerable interest, as e.g. properties of certain expansions of completely convex functions and, in particular, Schur's expansion of sin πx.

1

Let m be a non-negative integer. For a function $f \in \mathscr{C}^{2m}[0, 1]$ we introduce the functionals

$$A_0(f) = f(0), \tag{1a}$$

$$A_n(f) = (-1)^n \, (n!\,(n-1)!)^{-1} \int_0^1 f^{(2n)}(x)(x(1-x))^{n-1}\,(1-x)\,dx \qquad (1 \leqslant n \leqslant m); \tag{1b}$$

$$B_0(f) = f(1) - f(0), \tag{1c}$$

$$B_n(f) = (-1)^n \, (n!\,(n-1)!)^{-1} \int_0^1 f^{(2n)}(x)(x(1-x))^{n-1}\,(2x-1)\,dx \qquad (1 \leqslant n \leqslant m). \tag{1d}$$

By $\|g\|$ we denote the Chebyshev (maximum) norm of a function $g \in \mathscr{C}[0, 1]$. Recently, G. Berger and M. Tasche [2] have obtained the following remarkably useful result by appeal to the theory of right invertible operators:

0021-9045/91 $3.00

THEOREM (Berger and Tasche). (i) *If* $f \in \mathscr{C}^{2m}[0, 1]$, *then the Hermite–Lagrange 2-point interpolation polynomial, i.e., the (unique) polynomial p of degree (at most)* $2m+1$ *with*

$$p^{(j)}(0) = f^{(j)}(0), \qquad p^{(j)}(1) = f^{(j)}(1) \qquad (0 \leqslant j \leqslant m) \tag{2}$$

is given by

$$p = p_{m,f}(x) = \sum_{n=0}^{m} (a_n + b_n x)(x(1-x))^n, \tag{3a}$$

where

$$a_n = A_n(f), \qquad b_n = B_n(f) \qquad (0 \leqslant n \leqslant m). \tag{3b}$$

(ii) *If* $f \in \mathscr{C}^{\infty}[0, 1]$ *is a function with the property*

$$\lim_{n \to \infty} (2^{2n}(2n)!)^{-1} \, \| f^{(2n)} \| = 0 \tag{4}$$

then the series (3a) *with coefficients* (3b) *converges uniformly in* $[0, 1]$ *to* f.

The purpose of our note is to present a very simple proof of this theorem and so to provide an easy access to some of its consequences such as, for example, the positiveness of the coefficients in Schur's expansion of $\sin \pi x$ (cf. [2, 3]), which motivated the work of Berger and Tasche (for further examples and references see [2]). As regards the merits of their paper, it should be noted that formulae (3b) would hardly have been found by a method different from theirs.

Our proof is based on the following observation: From the relation (5) below it will become clear that it suffices to prove (3) for polynomials; and by the linearity of the functionals (1) it will even suffice to consider a suitable generating system (not necessarily a basis) of polynomials for which the verification of the identities (3) is a matter of easy calculation.

2

Proof of (i). Let there be $n \in N$, $h \in \mathscr{C}^{2n}[0, 1]$, g a polynomial of degree (at most) $2n-1$ with $g^{(j)}(0) = g^{(j)}(1) = 0$ ($0 \leqslant j \leqslant n-2$; this restriction is void if $n = 1$). We need the formula

$$\int_0^1 h^{(2n)}(x)\, g(x)\, dx = \sum_{j=0}^{n} (-1)^{j+1} \left(h^{(j)}(1)\, g^{(2n-j-1)}(1) - h^{(j)}(0)\, g^{(2n-j-1)}(0)\right) \tag{5}$$

which is easily established by repeated integration by parts.

An arbitrary polynomial p of degree (at most) $2m+1$ can be written in the form

$$p(x) = \sum_{n=0}^{m} (a_n + b_n x)(x(1-x))^n \tag{6a}$$

with unique coefficients, since the system of polynomials $q_k = (x(1-x))^k$, $r_k = xq_k$ $(k \geq 0)$ evidently forms a basis in the space of polynomials of arbitrary degree. We claim that these coefficients satisfy the relations

$$a_n = A_n(p), \qquad b_n = B_n(p) \qquad (n \geq 0). \tag{6b}$$

By the linearity of (1) it suffices to consider the elements of the basis introduced; we have to show that

$$A_n(r_n) = B_n(q_n) = 1, \qquad A_n(q_n) = B_n(r_n) = 0 \qquad (n \geq 0); \tag{7a}$$

$$A_n(r_k) = B_n(q_k) = 0, \qquad A_n(q_k) = B_n(r_k) = 0 \qquad (k \neq n; k, n \geq 0). \tag{7b}$$

If $n=0$, then (7a, b) follows at once on insertion of q_k, r_k in (1a, c). If $(0 \leq)$ $k < n$, then $q_k^{(2n)}$, $r_k^{(2n)}$ both vanish identically and (7b) follows trivially. If $1 \leq n < k$, then we combine (1) and (5) with $h = q_k$, resp. $h = r_k$, and $g = (x(1-x))^{n-1}$ $(1-x)$ or $g = (x(1-x))^{n-1}(2x-1)$, to obtain (7b) (note that here in each case $h^{(j)}(0) = h^{(j)}(1) = 0$ $(0 \leq j \leq n)$, since $h^{(j)}$ contains a factor $(x(1-x))^{k-j}$ with $k-j \geq k-n \geq 1$).

If $1 \leq n = k$, we obtain (7a) directly from (1b, d) by use of the obvious equalities $q_n^{(2n)} = (-1)^n (2n)!$, $r_n^{(2n)} = ((2n+1)x - n)\, q_n^{(2n)}$ and $\int_0^1 x^i(1-x)^j\, dx = i!j!(i+j+1)!)^{-1}$ $(i, j \geq 0)$.

From (5) we infer that the equalities

$$A_n(p) = A_n(f), \qquad B_n(p) = B_n(f) \qquad (0 \leq n \leq m) \tag{8}$$

hold for any two functions $p, f \in \mathscr{C}^{2m}[0, 1]$ with $p^{(j)}(0) = f^{(j)}(0)$, $p^{(j)}(1) = f^{(j)}(1)$ $(0 \leq j \leq m)$. Now (6) and (8) together imply that (3) indeed represents the H. L.-polynomial $p_{m,f}$ for f on $[0, 1]$. This proves (i).

3

Proof of (ii). Let $f \in \mathscr{C}^{\infty}[0, 1]$ be a function with property (4). Note that this condition entails the analogous property for odd derivatives: in fact, let x_0 be any element from $[0, 1]$. Take $x_1 \in [0, 1]$, with $|x_1 - x_0| = \frac{1}{2}$.

Then $((2n)!)^{-1}(x_1-x_0)^{2n}f^{(2n)}(x_2) = ((2n)!)^{-1}(x_1-x_0)^{2n}f^{(2n)}(x_0)+((2n+1)!)^{-1}(x_1-x_0)^{2n}f^{(2n+1)}(x_0)+((2n+2)!)^{-1}(x_1-x_0)^{2n+2}f^{(2n+2)}(x_3)$ with suitable numbers x_2, x_3 between x_0, x_1; hence

$$((2n+1)!\,2^{2n+1})^{-1}\,|f^{(2n+1)}(x_0)| \leqslant 2((2n)!\,2^{2n})^{-1}\,\|f^{(2n)}\|+((2n+2)!\,2^{2n+2})^{-1}\,\|f^{(2n+2)}\|.$$

Summarizing, we may take for granted that $(k!\,2^k)^{-1}\,\|f^{(k)}\|$ tends to 0 as $k\to\infty$. As a consequence, f can be expanded into a power series around any point $x_0\in[0,1]$, with radius of convergence $r=r(x_0)\geqslant 1-|x_0-\frac{1}{2}|$. Clearly f will be approximated by any such series uniformly in any closed interval contained in (x_0-r, x_0+r). In particular, this holds true for $x_0=\frac{1}{2}$, with $r\geqslant 1$. Splitting up the corresponding power series into an even and an odd part and applying the identity $(x-\frac{1}{2})^{2k}=(\frac{1}{4}-x(1-x))^k$, we find that f can be approximated by a series of the form $\sum(a_n+b_nx)(x(1-x))^n$ uniformly for $|x-\frac{1}{2}|\leqslant\frac{1}{2}$, that is, in $[0,1]$. But the partial sums p_m of this series are just the H.L.-polynomials of f in $[0,1]$, since $r_m^{(j)}(0)=r_m^{(j)}(1)=0$ $(0\leqslant j\leqslant m)$ evidently holds for the remainders $r_m=f-p_m$. This completes the proof of the theorem.

4

In this section we give a second proof of (6b). For the reasons mentioned it suffices to consider the two types of polynomials $p=s_k=x^k+(1-x)^k$ and $p=t_k=x^k-(1-x)^k$ $(k\geqslant 0)$. Note that the functions s_k are symmetric around $x=\frac{1}{2}$, and the functions t_k are antisymmetric. Consequently the system $\{s_k\}$, resp. $\{t_k\}$, generates the even, resp. odd, polynomials in the variable $x-\frac{1}{2}$.

First we establish the explicit expansions

$$s_k=\sum_{0\leqslant 2n\leqslant k} c_{kn}(x(1-x))^n,\qquad t_k=\sum_{0\leqslant 2n\leqslant k} d_{kn}(1-2x)(x(1-x))^n\qquad (k\geqslant 0)$$

where

$$c_{kn}=(-1)^n\left(\binom{k-n-1}{n-1}+\binom{k-n}{n}\right),\ d_{kn}=(-1)^n\left(\binom{k-n-1}{n-1}-\binom{k-n}{n}\right) \tag{9}$$

$(0\leqslant 2n\leqslant k;\ k\geqslant 0)$. We use the conventions $\binom{-1}{-1}=\binom{0}{0}=1$, $\binom{i}{-1}=0$ $(i\geqslant 0)$). From the obvious recursions $s_0=2$, $s_1=1$, $s_k=s_{k-1}-x(1-x)\,s_{k-2}$, resp. $t_0=0$, $t_1=-1+2x$, $t_k=t_{k-1}-x(1-x)\,t_{k-2}$ $(k\geqslant 2)$, we infer the recursions $c_{00}=2$, $d_{00}=0$; $c_{k0}=-d_{k0}=1$ $(k\geqslant 1)$; $c_{k1}=-k$, $d_{k1}=k-2$ $(k\geqslant 2)$; $c_{kn}=c_{k-1n}-c_{k-2n-1}$, $d_{kn}=d_{k-1n}-d_{k-2n-1}$ $(0<2n\leqslant k,\ k\geqslant 2)$. Now induction shows that the explicit expressions as given in (9) indeed share these recursions. Finally we apply the formula $\int_0^1 x^i(1-x)^j\,dx = i!j!\,((i+j+1)!)^{-1}$ to verify, by an easy calculation, the identities

$$c_{kn} = A_n(s_k),\quad 0 = B_n(s_k);\quad d_{kn} = A_n(t_k),\quad -2d_{kn} = B_n(t_k)\quad (0 \leqslant 2n \leqslant k,\ k \geqslant 0)$$

Our second proof of (6b) is complete.

We mention that C. Buchta [1] in a recent paper on a problem in geometric probability made use of the above expansion of s_k which he obtained as an (at first sight very special) consequence of the theorem. Here we have seen that indeed the general result can be derived from the special case.

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Printed in Belgium